PLANE NONSTATIONARY GAS FLOW WITH A STRONG DISCONTINUITY

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The problem of plane, nonstationary gas motion under the effect of a piston in the shape of a dihedral angle moving at constant velocity in the gas is considered. In contrast to one-dimensional motion under the effect of a flat piston, a curvilinear shockwave originates here, and the flow becomes nonisentropic and vortical. This problem is examined herein in a linear formulation when the angle of the piston breakpoint is assumed small. The linear problem reduces to an inhomogeneous Riemann–Hilbert problem whose solution is found explicitly. The problem under consideration adjoins a circle of problems associated with shockwave diffraction and reflection studied by Lighthill [1], Smyrl [2], Ter-Minassiants [3], etc.

1. Formulation of the Problem. A polytropic gas, at rest at $t \le 0$, is set in motion under the effect of the walls of a dihedral angle to which a constant velocity $\mathbf{U}_0 = (U_0, V_0)$ has been communicated at $t = 0$ so that the normal velocity components of the wall motion are directed toward the gas. A shockwave, whose front will be planar far from the vertex of the angle and curved in the domain of influence of the vertex, is formed ahead of the angular piston. It is required to compute the velocity and pressure fields in the region of influence of the vertex and, particularly, to determine the shockwave shape and the pressure on the piston.

Let us introduce a Cartesian rectangular X,Y coordinate system in the flow plane so that the origin would coincide with the vertex at $t = 0$, and the X axis would be directed along the axis of piston symmetry. Far from the vertex the flow is described by the known one-dimensional solution.

We seek the solution in the perturbed domain in the class of conical flows [4], by assuming all the desired functions $u°$, $v°$, $p°$, $\rho°$, $S°$ to be dependent on the variables $\xi = X/t$, $\eta = Y/t$. Here $\mathbf{u}° = (u°, v°)$ is the gas velocity, $p°$ is the pressure, $\rho°$ is the density, $S°$ is the entropy, and t is the time.

Let us introduce the new desired functions

$$U = u° - \xi, \quad V = v° - \eta, \quad P = p°(\xi, \eta), \quad R = \rho°(\xi, \eta), \quad S = S°(\xi, \eta) \tag{1.1}$$

The system of gasdynamics equations will reduce to the following:

$$(\mathbf{U}\cdot\nabla)\mathbf{U} + R^{-1}\nabla P + \mathbf{U} = 0, \quad \mathbf{U}\cdot\nabla R + R(\operatorname{div}\mathbf{U} + 2) = 0, \quad \mathbf{U}\cdot\nabla S = 0 \tag{1.2}$$

The perturbed flow domain is bounded by a line of degeneration of the type of the system (1.2) (AB and CD in Fig. 1)

$$|\mathbf{U}|^2 = U^2 + V^2 = C^2 \qquad (C^2 = \gamma R^{-1} P) \tag{1.3}$$

by the unknown shock front BC and the piston AED at the subsonic velocity V_0.

If the velocity V_0 is greater than the speed of sound, the vertex of the angle E would be outside the domain of ellipticity (given by the inequality $|\mathbf{U}| < C$) and a domain of hyperbolicity ($|\mathbf{U}| > C$) EDCF is added

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 45–50, May–June, 1971. Original article submitted December 15, 1970.

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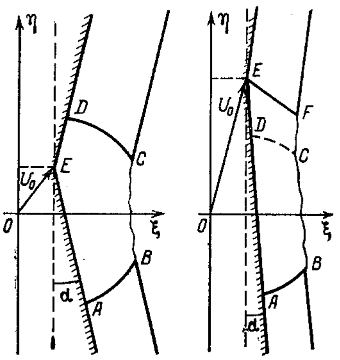

Fig. 1

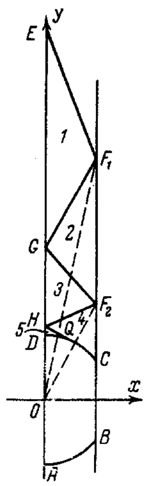

Fig. 2

to the domain of ellipticity ABCD; the line EF will be either a characteristic or the shock front depending on whether the piston aperture is greater or less than π.

The Hugoniot conditions, connecting the solution in the perturbed domain with known solutions in other domains, should be satisfied on the shock fronts, the condition of impenetrability should be satisfied on the piston, and the condition of continuous contact on all the remaining boundaries.

2. Equations and Boundary Conditions of the Linear Problem. Let the angle of the piston break α be small. Let us linearize the problem with respect to the small parameter α by taking the one-dimensional flow, which is obtained for $\alpha = 0$, as the fundamental solution. Let us represent the desired functions as follows:

$$U = U_0 - \xi + \alpha U_0 u, \quad V = -\eta + \alpha U_0 v, \quad P = p_1 + \alpha \rho_1 C_1 U_0 p, \quad R = \rho_1 (1 + \alpha\rho) \tag{2.1}$$

Here, p_1, ρ_1, C_1 are the gas parameters in the fundamental constant solution. After linearization, a linear system of equations is obtained for the dimensionless perturbations u, v, p, ρ considered as functions of the dimensionless variables x and y

$$(\mathbf{x}\cdot\nabla)\,\mathbf{u} = \nabla p$$
$$(\mathbf{x}\cdot\nabla)\,\rho = U_0\,(\operatorname{div}\mathbf{u})/C_1 \qquad \left(x = \frac{\xi - U_0}{C_1}, \quad y = \frac{\eta}{C_1}\right)$$
$$(\mathbf{x}\cdot\nabla)\,p = \operatorname{div}\mathbf{u} \tag{2.2}$$

Eliminating u and v from the last equation, we obtain an equation for the single function p

$$(x^2 - 1)p_{xx} + 2\,xy p_{xy} + (y^2 - 1)\,p_{yy} + 2xp_x + 2yp_y = 0 \tag{2.3}$$

The boundary conditions of the linear problem are obtained by linearizing the boundary conditions of the nonlinear problem and by carrying them over to the corresponding unperturbed boundaries. Let us write the equation of the perturbed portion of the shock front BC as

$$x = k + \alpha f(y) \qquad \left(k = \frac{D_0 - U_0}{C_1} = \left[\frac{(\gamma-1)M^2 + 2}{2\gamma M^2 - \gamma + 1}\right]^{1/2}, \; M = \frac{D_0}{C_0}\right) \tag{2.4}$$

Here, D_0 is the velocity of the unperturbed ($\alpha = 0$) shock, C_0 is the speed of sound in the gas at rest in front of the shock, γ is the adiabatic index. Using the Hugoniot relations on the shock, u, v, p, ρ can be calculated on the front (2.4)

$$u = LN^{-1}(f(y) - yf'(y)) \qquad \left(L = \frac{1}{2k}\,\frac{M^2+1}{M^2}\right)$$
$$v = -f'(y) \qquad \left(N = \frac{(\gamma+1)}{2}\,\frac{M^2 - 1}{(\gamma-1)M^2 + 2}\right)$$
$$p = N^{-1}(f(y) - yf'(y)), \qquad \rho = \frac{4}{(\gamma+1)kM^2}(f(y) - yf'(y)) \tag{2.5}$$

A relationship which the function p should satisfy on the boundary BC (BF) (Fig. 2)

$$(k^2 - 1)p_x + [(L + k)y - Nky^{-1}]p_y = 0 \tag{2.6}$$

follows from (2.5) and (2.2).

For the subsonic velocity V_0, i.e., when $V_0 < C_1$ (without limiting the generality it can be considered that $V_0 \geq 0$), the vertex E is within the domain of ellipticity.

From the impenetrability condition we obtain

$$u/_{AF} = V_0\,/\,U_0, \quad u/_{ED} = -\,V_0\,/\,U_0$$

Therefore, we have on the boundary AED

$$p_x = -Tk_1\delta(y - k_1) \qquad \left(T = \frac{k_1[(\gamma-1)M^2+2]}{k(M^2-1)},\ k_1 = \frac{V_0}{C_1}\right) \tag{2.7}$$

Here, $\delta(x)$ is the Dirac function. For $k_1 > 1$ the condition $p_x = 0$ is satisfied on ADE. The conditions of continuous contact with the known flow

$$p = p_2, \quad \rho = \rho_2, \quad u = u_2, \quad v = v_2$$

are satisfied on a line of degeneration of the type of (2.3) AB ($x^2 + y^2 = 1$, Fig. 2).

All these quantities are calculated by means of (2.5) if it is assumed that

$$f(y) = -y + \frac{\gamma+1}{2}k_1\frac{M^2}{M^2+1}$$

The continuous contact

$$p = -p_2, \quad \rho = -\rho_2, \quad u = -u_2, \quad v = -v_2$$

also holds on CD in the subsonic case.

For $k_1 > 1$ this same flow is in contact with the perturbed flow in the hyperbolic domain EFCD through the weak compression or rarefaction EF shock (depending on the sign of α). In addition to the conditions listed above, the condition of smoothness of the shock front at the points B and C

$$\int_{-k'}^{k'} \frac{p_y}{y}\,dy = -N^{-1}(1 + f'(k')) = K \qquad (x = k,\ k' = \sqrt{1-k^2}) \tag{2.8}$$

should be satisfied.

If $k_1 < 1$, then $f'(k') = 1$; if $k_1 > 1$, then $f'(k')$ is determined after the problem has been solved in the hyperbolic domain.

3. Solution of the Problem in the Hyperbolic Domain. It can be shown that the function p will be piecewise-constant in the hyperbolic domain. The domains of constant p will be connected through weak compression and rarefaction shocks. The rarefaction shocks originate from the linearization of certain rarefaction waves in the nonlinear problem. The fronts of these shocks coincide, in a first approximation, with the characteristics tangent to a unit circle. The following relationships

$$u_2 - u_1 = (p_2 - p_1)n_1,\ v_2 - v_1 = (p_2 - p_1)n_2,\ \rho_2 - \rho_1 = U_0C_1^{-1}(p_2 - p_1) \tag{3.1}$$

can be obtained from the Hugoniot conditions for weak shocks.

Here, $n = (n_1, n_2)$ is the normal to the characteristic on which the shock occurs in the linear approximation; the subscript 1 denotes the state in front of the shock, and the subscript 2 the state behind the shock. The quantity of shocks depends on the quantity k_1. All the weak shocks are easily computed by using the relationships (3.1).

As an illustration, let us consider the case pictured in Fig. 2, when there are five such shocks. The pressure p^1 on the discontinuity EF_1 is determined in domain 1 from the condition of impenetrability $u = V_0/U_0$ on EG. In domain 2 the desired functions should satisfy the Hugoniot conditions on F_1G and F_1F_2.

Such a solution can be constructed by introducing a contact discontinuity going, in a first approximation, along the straight contact characteristic F_1O of the fundamental solution. The quantity p^2 is determined from the condition on the contact discontinuity. Indeed, the function $f(y)$ is determined from (2.5) and the condition of passage of the front through the point F_1 by means of the known p^2. By means of the known $f(y)$ the u and v behind the contact discontinuity are determined. Ahead of the contact discontinuity u and v are expressed in terms of p^2 from conditions (3.1) on the shock F_1G. The condition of equal normal gas velocity components on F_1O yields an equation to determine p^2. We determine p^3 in domain 3 from the condition of impenetrability on the boundary GH, as in domain 1; hence, the conditions on the contact discontinuity F_1O are satisfied automatically. In domain 4 it is necessary to introduce a new contact discontinuity F_2O and to determine p^4 from the conditions on this contact discontinuity, as in domain 2, etc.

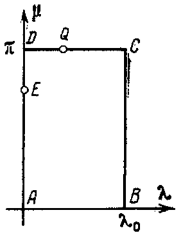

Fig. 3

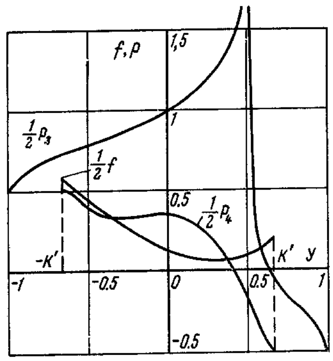

Fig. 4

After the computation in the hyperbolic domain, u, v, p, ρ are determined along CD. All these quantities have discontinuities at the point Q, and u, v, ρ at the points of intersection of the contact characteristics F_jO with the sonic circle. Let $p = p_*$ on DQ and $p = p^*$ on QC. The position of the point Q is given by the polar angle $\theta^* = \theta^*(k_1)$, measured from the x axis.

4. Finding the Solution in the Elliptic Domain. In the elliptic domain

$$r = \frac{2\beta}{\beta^2+1}, \qquad \theta = \theta' \begin{pmatrix} r^2 = x^2 + y^2 \\ \theta = \operatorname{arc\,tg}(y/x) \end{pmatrix} \tag{4.1}$$

is reduced to a Laplace equation by means of the Busemann–Chaplygin transformation. Let

$$\xi_1 = \beta\cos\theta, \quad \eta_1 = \beta\sin\theta, \quad \zeta = \xi_1 + i\eta_1$$

The conformal mapping

$$z = \ln\frac{1+\zeta}{1-\zeta} + \frac{1}{2}\pi i \tag{4.2}$$

maps the domain ABCD in the ζ plane into the rectangle

$$0 \leqslant \lambda \leqslant \lambda_0 = \frac{1}{2}\ln\frac{1+k}{1-k}, \qquad 0 \leqslant \mu \leqslant \pi$$

in the $z = \lambda + i\mu$ plane (Fig. 3).

Let us consider the analytic function

$$\Phi(z) = p_\lambda - ip_\mu \tag{4.3}$$

By virtue of the conditions of the boundary value problem, an inhomogeneous Hilbert problem originates with the discontinuous coefficients

$$a(z)p_\lambda - b(z)\,p_\mu = \varphi(z) \quad (z \in \Gamma) \tag{4.4}$$

Here, Γ is the contour ABCD and the coefficients a, b, and φ are given by the formulas

$$\begin{gathered}
\lambda = \lambda_0:\ \ a = \sin\mu\cos\mu, \quad b = Nk(1-k^2)^{-1} - L\cos^2\mu, \quad \varphi = 0 \\
\lambda = 0:\ a = 1, \quad b = 0, \quad \varphi = \begin{cases} -Tk_1(1-k_1^2)^{-1/2}\delta(\mu-\mu_0) & (k_1<1) \\ 0, \quad (k_1>1) \end{cases} \\
\mu = \pi:\ a = 1, \quad b = 0, \quad \varphi = \begin{cases} 0, \quad (k_1<1) \\ (p^*-p_2)\,\delta(\lambda-\lambda_1) & (k_1>1) \end{cases} \\
\mu = 0:\ \ a = 1, \quad b = 0, \quad \varphi = 0 \\
\left(\mu_0 = 2\operatorname{arc\,tg}\frac{1-\sqrt{1-k_1^2}}{k_1}, \quad \lambda_1 = \frac{1}{2}\ln\frac{1+\cos\theta^*}{1-\cos\theta^*}\right)
\end{gathered} \tag{4.5}$$

Since the coefficients of the Hilbert problem are discontinuous at the points B and C, the set of its solutions can be divided into classes of bounded or unbounded solutions at these points. Besides (4.4) and (4.5), it is necessary to impose additional conditions, the condition of smoothness of the shock front at the points B and C (2.8) and the condition of the variation of p along BC by a definite quantity

$$\lambda = \lambda_0:\ \int_0^\pi p_\mu\,d\mu = \begin{cases} -2p_2 & (k_1<1), \\ p^*-p_2 & (k_1>1), \end{cases} \quad \int_0^\pi \frac{p_\mu}{\cos\mu}\,d\mu = -k'K \tag{4.6}$$

The solution of the problem (4.4)-(4.6) in the class of functions bounded at the points of discontinuity of the coefficients is unique. To construct it, let us map the rectangle ABCD into the upper half plane by using the function

$$w(z) = \frac{\vartheta_2(0, q)\,\vartheta_2(-iz, q)}{\vartheta_3(0, q)\,\vartheta_3(-iz, q)} \qquad \left(q = \frac{1-k}{1+k}\right) \tag{4.7}$$

Here and henceforth, ϑ_1, ϑ_2, ϑ_3, ϑ_4 are elliptic theta functions [5]. The Hilbert problem in a half plane reduces in a known way to the Riemann problem [6]. The index of the Riemann problem turns out to equal one, i.e., the solution is determined to the accuracy of two arbitrary constants which are found uniquely from conditions (4.6). If the canonical solution X(w) is known for the corresponding homogeneous problem, then the solution of the inhomogeneous problem is written explicitly as

$$\Phi(w) = \frac{X(w)}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(\zeta)\, d\zeta}{(a - ib)\, X^+(\zeta)(\zeta - w)} + X(w)(B_1 w + D_1) \tag{4.8}$$

where B_1 and D_1 are arbitrary real constants, $X^+(\zeta)$ are the limit values of the function X(w) in the upper half plane. The solution of the problem therefore reduces to the construction of the canonical solution of the homogeneous problem. Let us represent X(w) as

$$X(w) = X_1(w) X_2(w) \tag{4.9}$$

where X_1(w) satisfies the condition on BC, and X_2(w) has a piecewise-constant argument on the boundary.

The first conditions (4.4), (4.5) can be written as [1]

$$\begin{gathered}\arg X(w(z)) = \operatorname{arc\,tg}(\alpha_1, \operatorname{tg}\mu) + \operatorname{arc\,tg}(\beta_1 \operatorname{tg}\mu) \\ \frac{1}{\alpha_1 + \beta_1} = L - N\frac{k}{1-k^2}, \qquad \frac{\alpha_1\beta_1}{\alpha_1 + \beta_1} = \frac{Nk}{1-k^2}\end{gathered} \tag{4.10}$$

Let us expand the right side of (4.10) in a Fourier sine series

$$\begin{gathered}\arg X(w(z)) = \pi - \sum_{n=1}^{\infty} \frac{(2 - a_1^n - b_1^n)}{n} \sin 2n\mu \\ a_1 = (\alpha_1 - 1) / (\alpha_1 + 1), \qquad b_1 = (\beta_1 - 1) / (\beta_1 + 1)\end{gathered} \tag{4.11}$$

Let us assume X_1(w) to equal

$$X_1(w(z)) = \exp\left\{-\sum_{n=1}^{\infty} (2 - a_1^n - b_1^n) \frac{\operatorname{ch} 2nz}{n \operatorname{sh} 2n\lambda_0}\right\} \tag{4.12}$$

The argument of this function on BC equals (4.11) to the accuracy of an additive constant. A mixed problem of the theory of functions is obtained to determine X_2(w), whose solution is given by the formula

$$X_2(w) = \frac{i}{\sqrt{1 - w^2}} = i\frac{\vartheta_3(0, q)\,\vartheta_3(-iz, q)}{\vartheta_4(0, q)\,\vartheta_4(-iz, q)} \tag{4.13}$$

We finally obtain

$$\begin{gathered}\Phi(z) = X(w(z))\left[-\frac{1}{\pi i}\frac{T_1}{w(i\mu_0) - w(z)} + B_1 w(z) + D_1\right] \qquad (k_1 < 1) \\ \Phi(z) = X(w(z))\left[\frac{1}{\pi i}\frac{T_2}{w(\lambda_1) - w(z)} + B_1 w(z) + D_1\right] \qquad (k_1 > 1)\end{gathered} \tag{4.14}$$

Here the function X(w(z)) is defined by (4.9), (4.12), (4.13) and

$$\begin{gathered}T_1 = \frac{Tk_1}{\sqrt{1 - k_1^2}} \frac{\vartheta_2(0, q)\,\vartheta_4^2(0, q)\,\vartheta_2(\mu_0, q)\,\vartheta_4(\mu_0, q)}{\vartheta_3(0, q)\,\vartheta_3^2(\mu_0, q)\,X^+(i\mu_0)} \\ T_2 = (p^* - p_*)\frac{\vartheta_2(0, q)\,\vartheta_2^2(0, q')\,\vartheta_2(\lambda_1, q')\,\vartheta_1(\lambda_1, q')}{\vartheta_3(0, q)\,\vartheta_3^2(\lambda_1, q')\,X^+(\lambda_1)} \qquad \left(q' = \exp\frac{\pi^2}{\ln q}\right)\end{gathered} \tag{4.15}$$

Calculating the limit values of Φ(z) on BC and separating out the imaginary part, we find p_μ, after which we determine the constants B_1 and D_1 from conditions (4.6).

Numerical computations were performed by means of the formulas obtained for different values of k and k_1. Represented in Fig. 4 are graphs of the functions $p_3 = p(0, y)$, the pressure on the piston, $p_4 =$

p(k, y), the pressure behind the shock on its front, and $f(y)$, which yields the shape of the shock computed for $k = 3/4$, $k_1 = 1/2$. As is seen from the figure, the pressure on the piston rises monotonically as the point E, the vertex of the dihedral angle, is approached. The function p has a logarithmic singularity at the point E.

After the function p has been determined in the elliptic domain, the functions u, v, ρ are found from (2.2) by quadratures.

The author is grateful to L. V. Ovsyannikov for interest in the research and useful comments.

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